DE GRUYTER
OPEN
10.1515/tvsb-2016-0027
Transactions of the VŠB – Technical University of Ostrava
Civil Engineering Series, Vol. 16, No. 2, 2016
paper #27

Hynek ŠTEKBAUER[1]

THE PULLEY ELEMENT

Abstract

The pulley is used in a number of structures for the mechanical advantage it gives. This paper presents an approach for the calculation of a pulley-cable system using a special pulley element in the finite element method. The Lagrange Multiplier method and Penalty method are used to define the pulley element, as described in this paper. Both approaches are easy to implement in general FEM codes.

Keywords

Pulley, finite element method, Penalty stiffness, Lagrange multiplier.

1 INTRODUCTION

Pulleys are used in many buildings and machine structures because of their structural simplicity and mechanical advantage due to transfer forces. A set of pulleys often creates comprehensive and complicated systems and therefore various pulley elements have been proposed for use in modelling [1, 2, 3, 4]. This paper presents a new algorithm that does not need to work with the stiffness of the cables connected to the pulley. As a result, the algorithm is easy to implement in general FEM codes. It is also possible to divide a cable between pulleys into multiple finite elements.

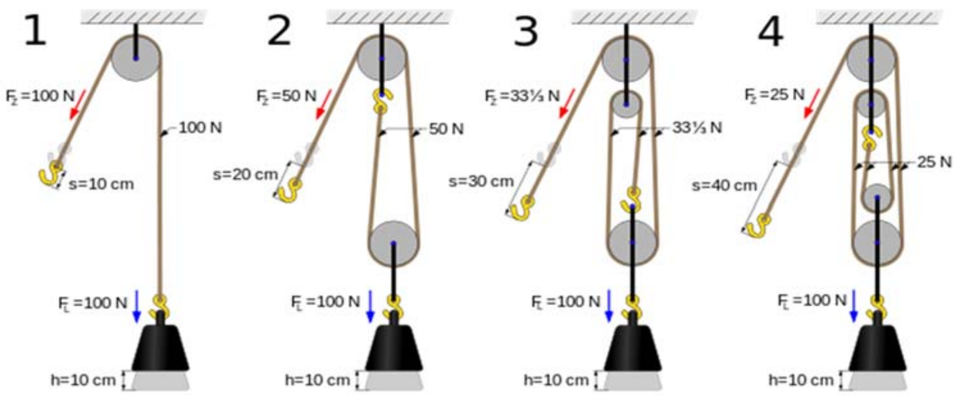

Fig. 1: Efficiency of pulley systems [1]

2 PULLEY ELEMENT DEFINITION

The presented idealized pulley element is defined by transformation matrixes of connected cables and three nodes (Fig. 2). The three nodes comprise two cable end nodes (N1, N3) and one node

[1] Ing. Hynek Štekbauer, Department of Structural Mechanics, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, Brno, 602 00, Czech Republic, phone: (+420) 541 147 378, e-mail: stekbauer.h@fce.vutbr.cz.

representing the pulley (N2). The radius of the pulley is neglected and all three nodes are at the same location.

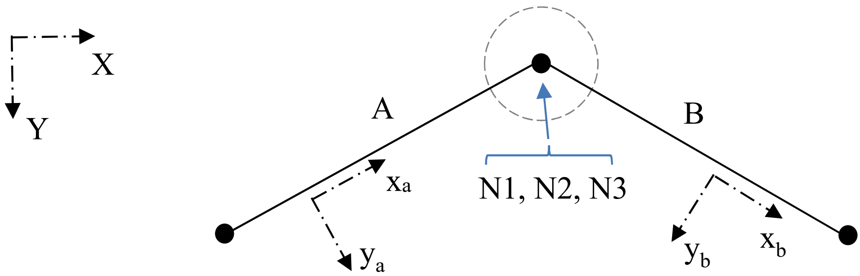

Fig. 2: Idealized pulley

The transformation matrix of each cable element is defined as

$$\boldsymbol{x}_a = \boldsymbol{T}_A \cdot \boldsymbol{X} \tag{1}$$

where

$$\boldsymbol{T}_A = \begin{bmatrix} T_{A11} & T_{A12} & T_{A13} \\ T_{A21} & T_{A22} & T_{A23} \\ T_{A31} & T_{A32} & T_{A33} \end{bmatrix}. \tag{2}$$

To describe the principle of the element we show the element from Fig. 2 with slightly shifted nodes N1, N2 and N3.

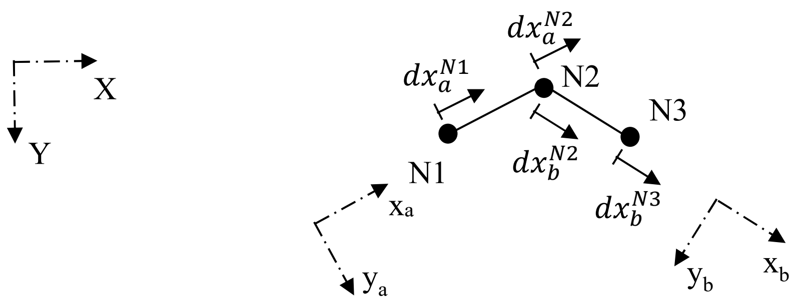

Fig. 3: Pulley element

The behaviour of the element is described by the equation

$$dx_a^{N1} - dx_a^{N2} = dx_b^{N3} - dx_b^{N2}. \tag{3}$$

To formulate Eq. 3 in the global coordinate system the following transformation is needed:

$$d\boldsymbol{X} = \boldsymbol{T}_A^T \cdot d\boldsymbol{x}_a. \tag{4}$$

Several constraints approaches can be used to implement Eq. 3 in the finite element method. Two of them are described in this paper.

2.1 The Lagrange Multiplier Method

The potential energy of the finite element model without constraints is

$$\Pi = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{f}. \tag{5}$$

The constraints imposed via Lagrange multipliers are stored in vector $\boldsymbol{\lambda}$ and form the Lagrangian.

$$L(\boldsymbol{d},\boldsymbol{\lambda}) = \Pi + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{d} - \boldsymbol{b}) = \frac{1}{2}\boldsymbol{d}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{d}^T\boldsymbol{f} + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{d} - \boldsymbol{b}). \tag{6}$$

By looking at the extreme of L with respect to $\boldsymbol{d}$ and $\boldsymbol{\lambda}$ one can obtain

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{d} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{b} \end{Bmatrix}. \tag{7}$$

The pulley element is defined as the combination of Eq.3-4 in $\boldsymbol{A}^T$

$$\boldsymbol{A}^T = \begin{bmatrix} T_{A11} \\ T_{A21} \\ T_{A31} \\ T_{B11} - T_{A11} \\ T_{B21} - T_{A21} \\ T_{B31} - T_{A31} \\ T_{B11} \\ T_{B21} \\ T_{B31} \end{bmatrix}, \tag{8}$$

where vector $\boldsymbol{b}$ is set to zeroes.

The method of Lagrange multipliers is exact, but also has disadvantages. The original system of equations needs to be expanded, and the detection of singularities is complicated because the system is not positive definite.

2.2 The Penalty Method

Due to the disadvantages of Lagrange multipliers, the Penalty stiffness method is also presented here. This method is easier to implement in FEM-based computer codes, but it has a main drawback which consists in the choice of penalty weight P. Constraint violation is dependent on the chosen penalty weight, rendering the Penalty method inexact. It can be shown that constraint violation is proportional to $1/P$ for sufficiently large P, but with larger P the stiffness matrix becomes more ill-conditioned.

The pulley element is defined as

$$\boldsymbol{K} = P \cdot \boldsymbol{A}^T \cdot \boldsymbol{A}. \tag{9}$$

3 VERIFICATION OF THE PULLEY ELEMENT

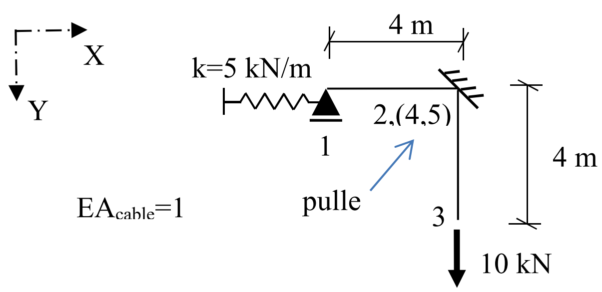

Fig. 4: Model

The model used to verify the pulley element is described in Fig.4. The support in node 1 has a stiffness of 5 kN/m in direction X, the support in node 2 is fixed in all directions, and node 3 is loaded by 10 kN in direction Y. The pulley is located in node 2, and internal nodes 4 and 5 were created to enable the pulley element to be used. The normal cross-section stiffness of the cables is 1 MN. The calculation was performed in the RFEM program.

Tab. 1: Results

Nodes	1		3	
Direction	dX	dY	dX	dY
Displacement LM	2.00 m	0.00 m	0.00 m	2.08 m
Displacement PS	2.00 m	0.00 m	0.00 m	2.08 m

The calculated displacements in Tab.1 are the same for both methods. The results correspond to the analytical solution.

4 CONCLUSIONS

A new algorithm for the calculation of pulleys using the finite element method is proposed. Pulley radius is neglected and a condition is defined for three nodes on each pulley (the pulley node and two ends of connected cables). Two methods of pulley element formulation are described, the Lagrange Multiplier method and the Penalty method, each with its advantages and disadvantages. The proposed element is used in an example calculation and has been found to be able to provide accurate results corresponding to those from an analytical solution.

ACKNOWLEDGMENT

This outcome has been achieved with the financial support of Brno University of Technology under project No. FAST-J-16-3558. The support is gratefully acknowledged.

LITERATURE

[1] AUFAURE, M. A finite element of cable passing through a pulley. Computers & Structures. Elsevier, 1993, Vol. 46, no.5, pp. 807-812. ISSN 0045-7949.

[2] JU, F. & CHOO, Y. Dynamic Analysis of Tower Cranes. Journal of Engineering Mechanics. American Society of Civil Engineers, 2005, Vol. 131, no.1, pp. 88-96. ISSN 0733-9399.

[3] JU, F. & CHOO, Y. Super element approach to cable passing through multiple pulleys. International Journal of Solids and Structures. Elsevier, 2005, Vol. 42, no.11-12, pp. 3533-3547. ISSN 0020-7683.

[4] Taylor, R., Mikulas Jr., M., Hedgepeth, J. A linearized discrete radius pulley model for finite element analysis, 35th Structures, Structural Dynamics, and Materials and Co-located Conferences, Hilton Head, SC, U.S.A., 1994, Vol. 6, pp. 2599-2606.